A SPIRAL HOLLOW JET OF AN IDEAL LIQUID

Yu. I. Petukhov

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 2, pp. 110-113, 1967

ABSTRACT: It is assumed in the theory of water outlets that the flow corresponding to the given head is the maximum possible [1] (the principle of maximum flow). The available experimental evidence only qualitatively reproduces the relations between the parameters implied by this principle, whereas numerical calculations agree very satisfactorily [2]. This indicates doubt as to whether the principle is exact. Here Moiseev's method [3] is used to ascertain the meaning of the relationships implied by the principle of maximum flow when used to calculate the potential flow of an ideal liquid in a centrifugal ejector. The resulting formula agrees well with experiment.

§1. Consider a spiral, axially symmetric flow of an ideal liquid as shown in Fig. 1 in axial section.

Here we have an xr coordinate system, in which x is the axis of symmetry of the flow, AA'G'F'FG is a section of the unbounded layer of liquid flowing in the cylindrical semi-infinite tube FF'E'E of unit radius, BB'C'C is the internal cylindrical wall, and BB'D'D is the free surface of the liquid, whose radius is $r = \sigma(x)$. The flow rate q and circulation Γ are given (we envisage potential flow, so Γ = const); v_r, v_φ, and v_x are the radial, tangential, and axial components of the velocity vector, respectively.

Since the flow is potential, $rv_\varphi = \Gamma/2\pi$; reduction in r increases the centrifugal force, since v_φ increases, which favors departure at the flow from the axis and formation of the cavity BB'D'D if q is not too large.

The internal semi-infinite wall BB'C'C has two functions related to two aspects of the problem that will be considered. Firstly, BB'C'C is a cylindrical tangent to the free surface at B and B' in the examination of the radius of the free surface as a function of q and Γ (case 1), and it assists in the computation.

Further, we have the question of the criterion for complete filling of the space between BB'C'C and FF'E'E by the spiral flow. In that case, BB'C'C is a real cylindrical wall of radius r_0 (case 2), and the point of detachment M moves onto BB'C'C and migrates towards x increasing as q increases. At some finite rate $q = q_0$, the entire space between the cylinders is filled.

Now $v_r = 0$ at M, while the axial velocity at M is v_0; and it varies from zero at B (when B and M coincide) to $q/\pi\,(1 - r_0^2)$ at infinity.

The Bernoulli integral at the free surface takes the following form (the liquid is weightless):

$$\frac{\Gamma^2}{4\pi^2\sigma^2} + v_r^2 + v_x^2 = \frac{\Gamma^2}{2\pi^2 r_0^2} + v_0^2 \,. \tag{1.1}$$

We have to put $v_0 = 0$ in (1.1) in considering case 1. In both cases it is assumed that $\sigma \to r_\infty$ = const for $x \to \infty$, i.e., we envisage asymptotically uniform flow.

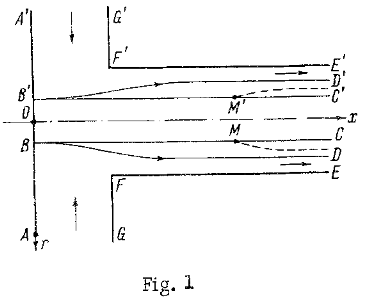

Fig. 1

§2. We now need a knowledge of the boundary derivative of the quasi-conformal mapping of a certain special form, namely we must find a function u that satisfies

$$\frac{\partial^2 u}{\partial\alpha^2} + \frac{\partial^2 u}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial u}{\partial\beta} = 0, \tag{2.1}$$

subject to the boundary conditions

$$u = 0 \ \text{ for } \ \beta = 1, \qquad u = n = \text{const} \ \text{ for } \ \beta = f(\alpha). \tag{2.2}$$

Function $f(\alpha)$ is smooth and differs little from a constant, so all of its derivatives are small, the largest ones being the first and second [3].

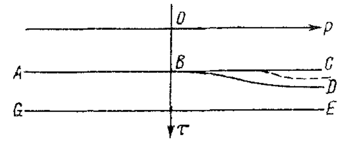

Fig. 2

We introduce in (2.2) the parameter ε defined by $\lambda = \varepsilon\alpha$:

$$\frac{\partial^2 u}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial u}{\partial\beta} + \varepsilon^2\frac{\partial^2 u}{\partial\lambda^2} = 0 \qquad \left(\varepsilon = \frac{\lambda}{\alpha}\right) \tag{2.3}$$

and seek a solution in the form of the series

$$u = u_0(\lambda, \beta) + \varepsilon^2 u_1(\lambda, \beta) + \ldots. \tag{2.4}$$

We substitute (2.4) into (2.3) and combine terms with the same powers of ε to get the following equations for the first two coefficients:

$$\frac{\partial^2 u_0}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial u_0}{\partial\beta} = 0, \quad \frac{\partial^2 u_1}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial u_1}{\partial\beta} = -\frac{\partial^2 u_0}{\partial\lambda^2}. \tag{2.5}$$

The solution to (2.5) subject to the boundary conditions applicable to (2.2) is

$$\begin{gathered} u_0 = a(\lambda)(\beta^2 - 1) \qquad \left(a(\lambda) = \frac{n}{\chi^2 - 1}\right) \\ u_1 = -\frac{a''}{4}\left[\frac{(\beta^2-1)^2}{4} - \beta^2(\ln\beta^2 - 1)\right] + \\ + \frac{a''\beta^2}{4(1-\chi^2)}\left[1 - \frac{(\chi^2-1)^2}{2} + (\chi^2\ln\chi^2 - \chi^2)\right] + \\ + \frac{a''}{4(1-\chi)^2}\left[\frac{(\chi^2-1)^2}{2} - \chi^2(\ln\chi^2 - 1)\right] \\ \left(\chi(\lambda) = f\left(\frac{\lambda}{\varepsilon}\right)\right). \end{gathered} \tag{2.6}$$

Taking only the two terms of (2.4) in (2.6), we find on the f curve that

$$\frac{1}{f^2}\left(\frac{\partial u}{\partial\beta}\right)^2 = \frac{4n^2}{(1-f^2)^2}\left\{1 + f''\frac{f}{f^2-1}\left(\frac{f^2-1}{2} - 1 + \frac{\ln f^2}{f^2-1}\right)\right\}. \tag{2.7}$$

Terms of higher powers of ε add to (2.7) terms containing higher derivatives of f, which are assumed to be small.

§3. We assume that we know the transformation $p = p(r, x)$, $\tau = \tau(r, x)$ that maps the rectangle ABCEFG in the xr-plane (Fig. 1) on a band in the $p\tau$-plane (Fig. 2), the corresponding points being clear from Figs. 1 and 2, in such a way that the points on the straight line GE in the $p\tau$-plane have an ordinate of unity, while the points on AC have an ordinate of r_0, and the equation of the form of (2.1) in the xr-plane has the same form in the $p\tau$-plane.

Let the curve $r = \sigma(x)$ become the curve $\tau = \theta(p)$. Taking the region ABDEG as being very similar to band ABCEG, we put

$$\sigma = r_0 + \Phi_1(p)\ (\theta - r_0) + \Phi_2(p)\ (\theta - r_0)^2 + \ldots, \tag{3.1}$$

in which Φ_1 and Φ_2 are certain functions of p that become infinite for p = 0. Moreover, the xr- and pτ-planes coincide for large x and p, so $\Phi_1 \to 1$ and $\Phi_2 \to 0$ as $p \to \infty$. In view of this, we will not use the third term in the expansion.

The tangential velocity is taken as being independent of x and dependent in a known fashion on r. We can introduce a complex potential W, whose components (the current function ψ and the potential φ) describe only radial and axial flow. The flow region in the W-plane is represented by a band, with $\psi = 0$ on GFE and $\psi = q/2\pi$ on ABD. Also,

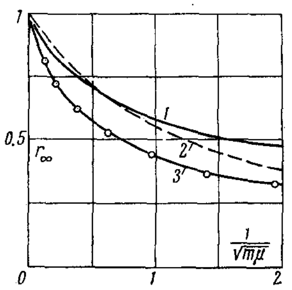

Fig. 3

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial x}, \quad v_x = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = 0.$$

Assuming that $v_r \ll v_x$ at the free surface, we write (1.1) in the form

$$\left|\frac{1}{\sigma}\frac{\partial\psi}{\partial r}\right|^2 + \frac{c^2}{\sigma^2} = \frac{c^2}{r_0^2} + v_0^2 \qquad \left(c = \frac{\Gamma}{2\pi}\right). \tag{3.2}$$

In the variables of the pτ-plane, this equation takes the form

$$\left|\frac{1}{\sigma}\frac{\partial\psi}{\partial\tau}\frac{\partial\tau}{\partial r}\right|^2 + \frac{c^2}{[r_0 + \Phi_1(\theta - r_0)^2]} = \frac{c^2}{r_0^2} + v_0^2. \tag{3.3}$$

The problem of finding ψ in the pτ-plane is analogous to that of §2, so we use (2.7) to get from (3.3) that

$$\frac{1}{(1 - m\delta^2)^2}\left\{1 + R(\delta)\frac{d^2\delta}{dp^2}\right\} =$$

$$= \mu\frac{N\Phi_1[2 + \Phi_1(\delta - 1)]}{[1 + \Phi_1(\delta - 1)]^2}(\delta - 1) + \nu N,$$

$$R(\delta) = \frac{m\delta}{m\delta^2 - 1}\left(\frac{m\delta^2 - 1}{2} - 1 + \frac{\ln m\delta^2}{m\delta^2 - 1}\right). \tag{3.4}$$

The notation in (3.4) is as follows:

$$N = \left(\frac{\partial r}{\partial\tau}\right)^2, \quad m = r_0^2, \quad \delta = \frac{\theta}{r_0}, \quad \mu = \frac{c^2\pi^2}{r_0^2 q^2},$$

$$\nu = v_0^2\frac{\pi^2}{q^2}, \qquad \begin{array}{l} N \to \infty \ \text{ for } \ p \to 0 \\ N \to 1 \ \text{ for } \ p \to \infty. \end{array}$$

The solution to (3.4) must satisfy the boundary conditions

$$\delta(0) = 1, \ d\delta / dp\,|_{p=0} = 0. \tag{3.5}$$

Now (3.4) contains the functions N and Φ_1, whose explicit form is unknown, since it is difficult to construct the quasi-conformal mapping; but if this can be done, (3.4) will include information on the effects of the conditions at the input (flow geometry) on the radius of the jet. In what follows we use only known limiting values of these functions.

§4. The μ and ν of (3.4) are unknown, but there is a condition for finding μ, since for a certain μ expression (3.4) must have a solution that tends asymptotically to a constant δ_∞. In case 2 let the detachment of the liquid occur at large x, so that we can put N = 1 and $\Phi_1 = 1$ in (3.4). The equation takes the form

$$\frac{1}{(1 - m\delta^2)^2}\left\{1 + R(\delta)\frac{d^2\delta}{dp^2}\right\} = \mu\frac{\delta^2 - 1}{\delta^2} + \nu. \tag{4.1}$$

As the point of detachment lies at large x, δ differs little from unity (the space between the cylinders is almost filled), and all derivatives of δ are small, so the squares of the derivatives may be neglected in comparison with the first powers.

We multiply (4.1) by $2m\delta\delta'$ and integrate it, subject to the boundary conditions of (3.5) to get

$$D(y)(y')^2 =$$
$$= (1/_2\mu k + 1/_3\mu)y^2 + y(k\nu - 1/_2\mu) + (l - \nu)$$
$$(y = \delta^2 - 1, \ k = m/(1 - m), \ l = 1/(1 - m)^2), \tag{4.2}$$

in which $y = \delta^2 - 1$ and D(y) is a function related to $R(\delta)$. From (4.2) we get

$$\int_0^y \frac{\sqrt{D}\,dy}{\sqrt{(1/_2\mu k + 1/_3\mu)y^2 + y(k\nu - 1/_2\mu) + (l - \nu)}} = p.$$

For $p \to \infty$ we have $y \to y_\infty$ = const, so this integral must diverge at $y = y_\infty$, which means that the expression under the radical has a repeated root y_∞. The condition for this is

$$(k\nu - 1/_2\mu)^2 - 4(l - \nu)(1/_2\mu k + 1/_3\mu) = 0. \tag{4.3}$$

If the point of detachment recedes to infinity, $\nu \to l$, and (4.3) gives

$$k\nu = 1/_2\mu, \quad \text{or} \quad \mu = 2m/(1 - m)^3. \tag{4.4}$$

The principle of maximum flow gives, for case 1, a formula similar in form to (4.4), with $m = r_0^2$ replaced by $m_1 = r_\infty^2$ [2]. The true meaning of (4.4) is, therefore, that this is the criterion for filling of the space between the cylinders by a spiral flow with the geometry of Fig. 1.

It is found by considering the two-dimensional analog of case 1 for a liquid with weight that N and Φ_1 have little effect on the result, so we consider (3.4) with $\nu = 0$, N = 1, and $\Phi_1 = 1$. We put $y = 1 - m\delta^2$ and expand $\ln m\delta^2$ as a series in y up to terms of the third power inclusive to get

$$\frac{1}{y} + \frac{y''}{6} - \mu\left(1 - \frac{m}{1 - y}\right)y = 0. \tag{4.5}$$

We multiply (4.5) by y', integrate, and proceed as in deriving (4.4) to get the condition for existence of an asymptotically homogeneous solution to (4.5):

$$\mu = \frac{1 + m}{(1 - m)^2}. \tag{4.6}$$

To this value of μ there corresponds $y_\infty = 1 - m$, i.e., expansion up to the third power in y corresponds to the assumption that the inner surface of the flow is cylindrical, since $y = 1 - m\delta^2$.

Figure 3 compares (4.4) and (4.6) with the experimental data of [2], in which curve 1 is from (4.4) and curve 2 from (4.6), while curve 3 is from experiment. Note that (4.6) was derived on the assumption that $r_0 = r_\infty$.

REFERENCES

1. G. N. Abramovich, Applied Gasdynamics [in Russian], Izd-vo TTL, 1950.

2. M. A. Gol'dshtik, A. K. Leont'ev, and I. I. Paleev, "Aerodynamics of a swirl chamber," Teploenergetika, no. 2, 1961.

3. N. N. Moiseev, "Asymptotic methods of the narrow-band type," collection: Problems in Mathematics and Mechanics [in Russian], Izd-vo AN SSSR, 1961.

25 January 1966 Novosibirsk